Defining Stable Homotopy

Riley Moriss

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1 Excision for Homotopy Groups

Homotopy groups will have an excision property in a *range* determined by their connectivity. Recall that n-connected means that the $i \leq n$th homotopy groups are trivial.

Theorem. *Let X be a CW complex decomposed as the union of subcomplexes A and B such that the intersection of A and B is both non-empty and connected. If $(A, A \cap B)$ is $m-$connected and $(B, A \cap B)$ is n-connected for $m, n \geq 0$ then for the inclusion $i : (A, A \cap B) \to (X, B)$ we get that*

$$i_* : \pi_k(A, A \cap B) \to \pi_k(X, B)$$

is an isomorphism for $k < m + n$ and a surjection for $k = m + n$.

Theorem. *For an r connected CW pair (X, A) where A is s-connected for $r, s \geq 0$ the map induced by the quotient $X \mapsto X/A$*

$$\pi_i(X, A) \to \pi_i(X/A)$$

is an iso for $i \leq r + s$ and a surjection for $i = r + s + 1$.

get an idea of how the proof works

2 Freudenthals Suspension Theorem

The Freudenthal suspension theorem is a consequence of excision for homotopy groups

Theorem. *The suspension map induces an isomorphism on homology groups*

$$\pi_i(X) \to \pi_{i+1}(SX)$$

for $i < 2n - 1$ for an $n - 1$ connected CW complex X.

Decompose the suspension SX into the union of two cones C_+X, C_-X which intersect at a copy of X. The suspension map is

Is this just an inclusion? Why what is the suspension map...?

$$\pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X)$$

The cone is contactable and so $\pi_{i+1}(C_\pm X, X) \cong \pi_i(X)$, note that the degree is lowered. For the same reason $\pi_{i+1}(SX, C_- X) \cong \pi_{i+1}(SX)$. They can also be deduced from the LES.

Now from the LES of $C_\pm X, X$ we get

$$\pi_i(X) \to \pi(C_\pm) \to \pi_i(C_\pm, X)$$

which by contractability of the cone gives

$$0 \to \pi_i(C_\pm, X) \to \pi_{i-1}(X) \to 0$$

hence if X is $n-1$ connected then $(C_\pm, X)$ is n connected. Now apply excision to see that the inclusion is an iso.

hypothesis for excision is that the intersection, which here is X is connected? Whats going on here... Might need to assume that X is connected.

3 Stable Homotopy

If X is n connected CW complex then because $n < 2n+1$ we get by Freudenthal's theorem that $0 = \pi_n(X) \cong \pi_{n+1}(\Sigma X)$, hence SX is $n+1$ connected. We want to see why the sequence below stabilises *for any X*

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \cdots \to \pi_{i+k}(\Sigma^k X) \to \cdots$$

Now assume that X is -1 connected, that is no assumption at all. Then we know that $\Sigma^k X$ is $k-1$-connected, applying Fruedenthal tells us that for $j < 2(k-1)+1 = 2k-1$

$$\pi_j(\Sigma^k X) \cong \pi_{j+1}(\Sigma^{k+1} X)$$

For a given i we observe that for a large k we have that $i + k < 2k - 1$, hence the maps in the above sequence become isomorphisms.

4 Homotopy Groups of Spheres

Using cellular approximation for maps we can see that $\pi_n(S^k) = 0$ for all $n < k$. *Little spheres into bigger spheres are contractable.* The idea is simply to homotope to a cellular map. The CW structure of the k sphere is to glue a k disk and glue the boundary to a point. Hence a map (D^n, S^{n-1}) into S^k for $n < k$ requires that $f(D^n) \subseteq (S^k)^n = \{*\}$ that is the n skeleton of the k sphere, which for $n < k$ is just a point.

In general we dont know what the homotopy groups are for $n \geq k$, because we dont have (full) excision it is even harder to compute. Instead we deal with the stable groups which do form a homology. In particular the hypotheses of Fruednethal suspension theorem applys to spheres and so we get that s

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

for $i < 2n - 1$. Thus we have that for each r

$$\pi_{k_1+r}(S^{k_1}) \cong \pi_{k_2+r}(S^{k_2})$$

where $k_i >> r$ (specifically $k \geq r+2$). This group is called the r-stem.

$$\pi_r^S \quad := \quad \pi_{r+k}(S^k) = \pi_r(S^0) = \mathrm{colim}_k \pi_{r+k}(S^k)$$

We can see this in the following diagram, where Σ is the suspension map (not necissarily iso until high degrees), suspending domain and codomain.

$$
\begin{tikzcd}
\pi_j(S^i) & j & & & & \\
i & \pi_0(S^0) \arrow[dr, "\Sigma"] & \pi_1(S^0) \arrow[dr, "\Sigma"] & \pi_2(S^0) & \cdots & \pi_n(S^0) \\
 & \pi_0(S^1) = 0 & \pi_1(S^1) \arrow[dr, "\Sigma"] & \pi_2(S^1) \arrow[ddrr] & & ? \\
 & \pi_0(S^2) = 0 & 0 & \pi_2(S^2) \arrow[ddrr] & \ddots & \\
 & \vdots & & \ddots & \ddots & \pi_{n-1}(S^n) \\
 & \pi_n(S^0) = 0 & 0 & & 0 & \pi_n(S^n)
\end{tikzcd}
$$

So the stable groups are the diagonals. The top right corner is a mystery (not a homology theory), the bottom left is trivial but the diagonals are in progress (they are a homology theory). For spheres in particular the diagonals are indexed by the top row, becuase the left column will index sequences of zeroes so forget about it.

References