## Defining Stable Homotopy

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#### 1 Excision for Homotopy Groups

Homotopy groups will have an excision property in a *range* determined by their connectivity. Recall that *n*-connected means that the  $i \leq n$ th homotopy groups are trivial.

**Theorem.** Let X be a CW complex decomposed as the union of subcomplexes A and B such that the intersection of A and B is both non-empty and connected. If  $(A, A \cap B)$  is m-connected and  $(B, A \cap B)$  is n-connected for  $m, n \ge 0$  then for the inclusion  $i : (A, A \cap B) \to (X, B)$  we get that

 $i_*: \pi_k(A, A \cap B) \to \pi_k(X, B)$ 

is an isomorphism for k < m + n and a surjection for k = m + n.

**Theorem.** For an r connected CW pair (X, A) where A is s-connected for  $r, s \ge 0$  the map induced by the quotient  $X \mapsto X/A$ 

 $\pi_i(X, A) \to \pi_i(X/A)$ 

is an iso for  $i \leq r + s$  and a surjection for i = r + s + 1.

### 2 Freudenthals Suspension Theorem

The Freudenthal suspension theorem is a consequence of excision for homotopy groups

Theorem. The suspension map induces an isomorphism on homology groups

$$\pi_i(X) \to \pi_{i+1}(SX)$$

for i < 2n - 1 for an n - 1 connected CW complex X.

Decompose the suspension SX into the union of two cones  $C_+X, C_-X$  which intersect at a copy of X. The suspension map is

$$\pi_{i+1}(C_+X,X) \to \pi_{i+1}(SX,C_-X)$$

Is this just an inclusion? Why what is the suspension map...?

get an idea of how the proof works The cone is contactable and so  $\pi_{i+1}(C_{\pm}X, X) \cong \pi_i(X)$ , note that the degree is lowered. For the same reason  $\pi_{i+1}(SX, C_X) \cong \pi_{i+1}(SX)$ . They can also be deduced from the LES.

Now from the LES of  $C_{\pm}X, X$  we get

$$\pi_i(X) \to \pi(C_{\pm}) \to \pi_i(C_{\pm}, X)$$

which by contractability of the cone gives

$$0 \to \pi_i(C_{\pm}, X) \to \pi_{i-1}(X) \to 0$$

hence if X is n-1 connected then  $(C_{\pm}, X)$  is n connected. Now apply excision to see that the inclusion is an iso.

#### 3 Stable Homotopy

If X is n connected CW complex then because n < 2n + 1 we get by Freudenthal's theorem that  $0 = \pi_n(X) \cong \pi_{n+1}(\Sigma X)$ , hence SX is n + 1 connected. We want to see why the sequence below stabilises for any X

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \dots \to \pi_{i+k}(\Sigma^k X) \to \dots$$

Now assume that X is -1 connected, that is no assumption at all. Then we know that  $\Sigma^k X$  is k-1-connected, applying Fruedenthal tells us that for j < 2(k-1) + 1 = 2k - 1

$$\pi_j(\Sigma^k X) \cong \pi_{j+1}(\Sigma^{k+1} X)$$

For a given i we observe that for a large k we have that i + k < 2k - 1, hence the maps in the above sequence become isomorphisms.

#### 4 Homotopy Groups of Spheres

Using cellular approximation for maps we can see that  $\pi_n(S^k) = 0$  for all n < k. Little spheres into bigger spheres are contractable. The idea is simply to homotope to a cellular map. The CW structure of the k sphere is to glue a k disk and glue the boundary to a point. Hence a map  $(D^n, S^{n-1})$  into  $S^k$  for n < k requires that  $f(D^n) \subseteq (S^k)^n = \{*\}$  that is the n skeleton of the k sphere, which for n < k is just a point.

In general we dont know what the homotopy groups are for  $n \ge k$ , because we dont have (full) excision it is even harder to compute. Instead we deal with the stable groups which do form a homology. In particular the hypotheses of Fruednethal suspension theorem applys to spheres and so we get that s

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

for i < 2n - 1. Thus we have that for each r

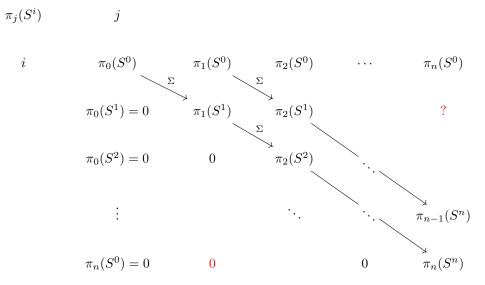
$$\pi_{k_1+r}(S^{k_1}) \cong \pi_{k_2+r}(S^{k_2})$$

where  $k_i >> r$  (specifically  $k \ge r+2$ ). This group is called the *r*-stem.

$$\pi_r^S := \pi_{r+k}(S^k) = \pi_r(S^0) = \operatorname{colim}_k \pi_{r+k}(S^k)$$

We can see this in the following diagram, where  $\Sigma$  is the suspension map (not necissarily iso until high degrees), suspending domain and codomain.

hypothesis for excision is that the intersection, which here is X is connected? Whats going on here... Might need to assume that X is connected.



So the stable groups are the diagonals. The top right corner is a mystery (not a homology theory), the bottom left is trivial but the diagonals are in progress (they are a homology theory). For spheres in particular the diagonals are indexed by the top row, because the left column will index sequences of zeroes so forget about it.

# References